

关于 $C_8 \oplus C_8$ 中极小零和子列的一个性质

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摘要: 如果在群 $C_n \oplus C_n$ 中, 每个含有 $2n-1$ 个元素的极小零和序列中都包含一些阶数为 $n-1$ 的元素, 那么我们称正整数 n 具有性质 B。在二维阿贝尔群的零和理论中, 性质 B 是一个中心议题。关于性质 B 这一问题最早是由高维东教授和 A. Geroldinger 提出并进行研究^[1-3]。之后, 他们证明了如果 n 具有性质 B^[4-6], 当 n 大于等于 6 时, $2n$ 也具有性质 B; 还证明了如果 $n \in \{2, 3, 4, 5, 6, 7\}$, n 具有性质 B。在文 [7] 中, 我们证明了 $n=10$ 时, n 具有性质 B。本文证明 $n=8$ 时, n 也具有性质 B。

关键词: 阿贝尔群; 零和子列; 性质 B

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A Property on Minimal Zero-sum Subsequence in $C_8 \oplus C_8$

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Abstract: We say a positive integer n has Property B if every minimal zero-sum subsequence of $2n-1$ elements in $C_n \oplus C_n$ contains some elements $n-1$ times. Property B is a central topic in zero-sum theory on abelian group G with rank two. Property B has been first formulated and investigated by professor W. D. Gao and A. Geroldinger in^[1-3]. It has been proved that if $n \geq 6$ and if n has Property B, then $2n$ has Property B. It has been also proved that if $n \in \{2, 3, 4, 5, 6, 7\}$, then n has property B^[4-6]. In [7], we proved that $n=10$ has Property B. In this paper, we will verify that $n=8$ has Property B.

Keywords: abelian group; zero-sum subsequence; Property B

0 Introduction

Let N denote the set of positive integers and $N_0 = N \cup \{0\}$. Let Z denote the set of integers. For $a, b \in Z$ with $a \leq b$, we define^[8-10] $[a, b] = \{x \in Z \mid a \leq x \leq b\}$.

Let G be a finite abelian group. Let $F(G)$ be the free abelian monoid, multiplicatively written, with basis G . The elements of $F(G)$ are called sequences over G . We write sequence $S \in F(G)$ in the form

$$S = \prod_{g \in G} g^{V_g(S)},$$

with $V_g(S) \in N_0$ for all $g \in G$.

We call $V_g(S)$ the multiplicity of g in S . We say

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that S contains g , if $V_g(S) > 0$, and g is called a term of S . The unit element $1 \in F(G)$ is called the empty sequence. A sequence S_1 is called a subsequence of S if $S_1 | S$ in $F(G)$ (equivalently, $V_g(S_1) \leq V_g(S)$ for all $g \in G$), and it is called a proper subsequence of S if it is a subsequence with $1 \neq S_1 \neq S$. Let $S_1, S_2 \in F(G)$, we denote by $S_1 S_2$ the sequence

$$\prod_{g \in G} g^{V_g(S_1) + V_g(S_2)} \in F(G).$$

If a sequence $S \in F(G)$ is written in the form $S = g_1 \cdots g_l$, we tacitly assume that $l \in N_0$ and $g_1, \cdots, g_l \in G$. For a sequence

$$S = g_1 \cdots g_l = \prod_{g \in G} g^{V_g(S)} \in F(G)$$

We call $|S| = l = \sum_{g \in G} V_g(S) \in N_0$ the length of S ,

$$h(S) = \max\{V_g(S) \mid g \in G\} \in \mathbb{N}, |S|$$

the maximum of the multiplicities of S ,

$$\sigma(S) = \sum_{i=1}^l g_i = \sum_{g \in G} V_g(S) g \in G$$

the sum of S ,

$$\Sigma(S) = \{ \sum_{i \in I} g_i \mid I \subset \{1, \dots, l\} \text{ with } 1 \leq |I| \leq l \}$$

the set of all subsums of S .

The sequence S is called

- zero-sum free if $0 \notin \Sigma(S)$,
- a zero-sum sequence if $\sigma(S) = 0$,
- a minimal zero-sum sequence if it is a non-empty zero-sum sequence and every proper subsequence is zero-sum free,
- a short zero-sum sequence if it is a zero-sum sequence of length

$$|S| \in \mathbb{N}, \exp(G).$$

We denote by $D(G)$ the smallest integer $l \in N$ such that every sequence $S \in F(G)$ of length $|S| \geq l$ has a non-empty zero-sum subsequence.

The invariant $D(G)$ is called the Davenport constant [1] of G .

Let C_n be the cyclic group of order n . Let $n \geq 2$ be a positive integer. We say that n has Property B if every minimal zero-sum sequence in $F(C_n \oplus C_n)$ of length $2n - 1$ contains some elements with multiplicity $n - 1$. It has been conjectured that.

Conjecture. Every positive integer $n \geq 2$ has Property B (for e. g., see [2] and [3]).

In this paper, we obtain that $n = 8$ has Property B.

1 Main result

Theorem. Positive integer $n = 8$ has property B.

Proof. let $S \in F(C_8 \oplus C_8)$ be a minimal zero sum sequence with length 15. We want to prove that there exists at least an element g in S such that $V_g(S) \geq 7$.

In [4] and [5], it is proved that there exists a homomorphism φ from $C_8 \oplus C_8$ to $C_4 \oplus C_4$, such that

$$\varphi(S) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \prod_{i=1}^{15-l} \begin{pmatrix} 1 \\ a_i \end{pmatrix}$$

where

$$l \equiv 3 \pmod{4}, \sum_{i=1}^{15-l} a_i \equiv 1 \pmod{4} \text{ and } \ker \varphi \cong C_2 \oplus C_2.$$

One can factor $\varphi(S)$ into

$$\varphi(S) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^l \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{N_0} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{N_1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}^{N_2} \begin{pmatrix} 1 \\ 3 \end{pmatrix}^{N_3}$$

where $N_0 + N_1 + N_2 + N_3 = 15 - l$.

We set $S = TT_0T_1T_2T_3$ such that

$$\varphi(T) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^l, \varphi(T_i) = \begin{pmatrix} 1 \\ i \end{pmatrix}^{N_i} \quad (i = 0, 1, 2, 3)$$

Let

$$T = \prod_{i=1}^l a_i, T_0 = \prod_{i=1}^{N_0} x_i, T_1 = \prod_{i=1}^{N_1} y_i,$$

$$T_2 = \prod_{i=1}^{N_2} z_i \text{ and } T_3 = \prod_{i=1}^{N_3} t_i$$

$(a_i, x_i, y_i, z_i, t_i \in C_8 \oplus C_8)$, without loss of the generality, we consider that N_0 is the maximum one among N_i ($i = 0, 1, 2, 3$).

We distinguish three cases.

Case 1. Suppose that $l = 3$.

Through discussion, we know that there exist 30 different values of (N_0, N_1, N_2, N_3) . Now we only discuss the following four values of (N_0, N_1, N_2, N_3) .

Subcase 1.

$$(N_0, N_1, N_2, N_3) = (7, 2, 2, 1).$$

Whence

$$\varphi(S) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^7 \begin{pmatrix} 1 \\ 1 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and $S = TT_0T_1T_2T_3$, where

$$T = \alpha_1 \alpha_2 \alpha_3, T_0 = x_1, \dots, x_7,$$

$$T_1 = y_1 y_2, T_2 = z_1 z_2, T_3 = t.$$

We can assert that there exist at most two distinct elements in T_0 . So we set that

$$T_0 = x_1^r x_2^{7-r} \quad (4 \leq r \leq 7).$$

If $r = 7$, theorem is true.

If $4 \leq r < 7$, let

$$h_1 = 4x_1,$$

$$h_2 = 3x_1 + x_2,$$

$$h_3 = x_1 + y_1 + y_2 + z_1$$

and

$$h_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2x_1 + z_2 + t,$$

they are in $\ker \varphi$ and $h_3 = h_4$. We can obtain that $h_3 + h_4 = 0$, which implies that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 3x_1 + y_1 + y_2 + z_1 + z_2 + t = 0,$$

a contradiction.

Subcase 2.

$$(N_0, N_1, N_2, N_3) = (6, 1, 3, 2)$$

whence

$$\varphi(S) = \binom{0}{1}^3 \binom{1}{0}^6 \binom{1}{1} \binom{1}{2}^3 \binom{1}{3}^2$$

and

$$S = TT_0 T_1 T_2 T_3,$$

where $T = \alpha_1 \alpha_2 \alpha_3$, $T_0 = x_1, \dots, x_6$, $T_1 = y$, $T_2 = z_1 z_2 z_3$ and $T_3 = t_1 t_2$.

Subcase 3.

$$(N_0, N_1, N_2, N_3) = (5, 1, 2, 4)$$

whence

$$\varphi(S) = \binom{0}{1}^3 \binom{1}{0}^5 \binom{1}{1} \binom{1}{2}^2 \binom{1}{3}^4$$

and

$$S = TT_0 T_1 T_2 T_3,$$

where $T = \alpha_1 \alpha_2 \alpha_3$, $T_0 = x_1, \dots, x_5$, $T_1 = y$, $T_2 = z_1 z_2$ and $T_3 = t_1 t_2 t_3 t_4$.

Subcase 4.

$$(N_0, N_1, N_2, N_3) = (4, 2, 3, 3)$$

whence

$$\varphi(S) = \binom{0}{1}^3 \binom{1}{0}^4 \binom{1}{1}^2 \binom{1}{2}^3 \binom{1}{3}^3$$

and

$$S = TT_0 T_1 T_2 T_3,$$

where $T = \alpha_1 \alpha_2 \alpha_3$, $T_0 = x_1, \dots, x_4$, $T_1 = y_1 y_2$, $T_2 = z_1 z_2 z_3$ and $T_3 = t_1 t_2 t_3$.

For subcases 2–4, We also assert that there exist at most two distinct elements in T_0 .

Case 2. Suppose that $l = 7$. We have that

$$\varphi(S) = \binom{0}{1}^7 \prod_{i=1}^8 \binom{1}{a_i}$$

and $S = TT'$, where $T = \alpha_1, \dots, \alpha_7$ and $T' = x_1, \dots, x_8$, such that

$$\varphi(T) = \binom{0}{1}^7 \text{ and } \varphi(T') = \prod_{i=1}^8 \binom{1}{a_i}.$$

Case 3. Suppose that $l = 11$.

Whence

$$\varphi(S) = \binom{0}{1}^{11} \prod_{i=1}^4 \binom{1}{a_i}$$

and $S = TT'$, where $T = \alpha_1 \cdots \alpha_{11}$, $T' = x_1 \cdots x_4$, such that

$$\varphi(T) = \binom{0}{1}^{11} \text{ and } \varphi(T') = \prod_{i=1}^4 \binom{1}{a_i}.$$

Similar to the proof of case 1, we can verify that there are at most two distinct elements in T of case 2 and case 3.

Theorem is true.

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